HL Paper 1

Find the values of x for which the vectors $\begin{pmatrix} 1 \\ 2\cos x \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 2\sin x \\ 1 \end{pmatrix}$ are perpendicular, $0 \leqslant x \leqslant \frac{\pi}{2}$.

a. Given that $\arctan\left(\frac{1}{5}\right) + \arctan\left(\frac{1}{8}\right) = \arctan\left(\frac{1}{p}\right)$, where $p \in \mathbb{Z}^+$, find p. [3]

b. Hence find the value of $\arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{5}\right) + \arctan\left(\frac{1}{8}\right)$. [3]

a. Use the identity $\cos 2\theta = 2\cos^2\theta - 1$ to prove that $\cos \frac{1}{2}x = \sqrt{\frac{1+\cos x}{2}}$, $0 \leqslant x \leqslant \pi$. [2]

b. Find a similar expression for $\sin \frac{1}{2}x$, $0 \leqslant x \leqslant \pi$. [2]

c. Hence find the value of $\int_0^{\frac{\pi}{2}} \left(\sqrt{1+\cos x} + \sqrt{1-\cos x}\right) \mathrm{d}x$. [4]

a. Show that $\frac{\sin 2\theta}{1+\cos 2\theta} = \tan\theta$. [2]

b. Hence find the value of $\cot \frac{\pi}{8}$ in the form $a + b\sqrt{2}$, where $a, b \in \mathbb{Z}$. [3]

a. Show that $\cot\alpha = \tan\left(\frac{\pi}{2} - \alpha\right)$ for $0 < \alpha < \frac{\pi}{2}$. [1]

b. Hence find $\int_{\tan\alpha}^{\cot\alpha} \frac{1}{1+x^2} \mathrm{d}x$, $0 < \alpha < \frac{\pi}{2}$. [4]

In the triangle ABC, $\mathrm{AB} = 2\sqrt{3}$, AC = 9 and $\mathrm{B\hat{A}C} = 150^\circ$.

a. Determine BC, giving your answer in the form $k\sqrt{3}$, $k \in \mathbb{Z}^+$. [3]

b. The point D lies on (BC), and (AD) is perpendicular to (BC). Determine AD. [4]

Let $z = 1 - \cos 2\theta - \mathrm{i}\sin 2\theta,\ z \in \mathbb{C},\ 0 \leqslant \theta \leqslant \pi.$

a. Solve $2\sin(x + 60^\circ) = \cos(x + 30^\circ),\ 0^\circ \leqslant x \leqslant 180^\circ.$ [5]

b. Show that $\sin 105^\circ + \cos 105^\circ = \frac{1}{\sqrt{2}}.$ [3]

c.i. Find the modulus and argument of z in terms of θ. Express each answer in its simplest form. [9]

c.ii. Hence find the cube roots of z in modulus-argument form. [5]

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a. Find the value of $\sin\frac{\pi}{4} + \sin\frac{3\pi}{4} + \sin\frac{5\pi}{4} + \sin\frac{7\pi}{4} + \sin\frac{9\pi}{4}.$ [2]

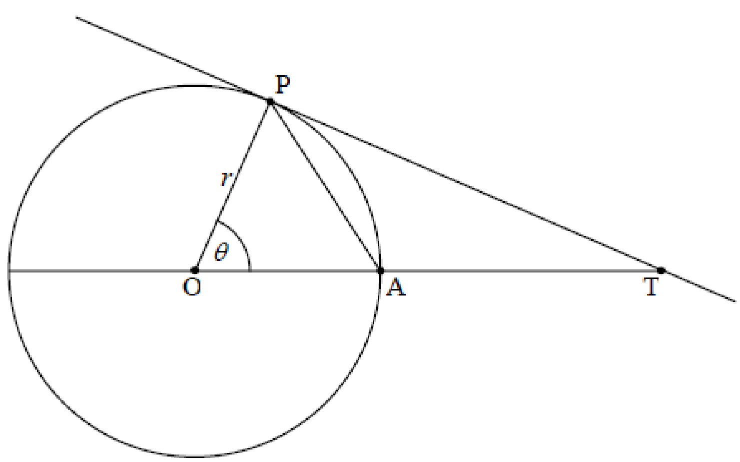
b. Show that $\frac{1-\cos 2x}{2\sin x} \equiv \sin x,\ x \neq k\pi$ where $k \in \mathbb{Z}$. [2]

c. Use the principle of mathematical induction to prove that [9]

$\sin x + \sin 3x + \ldots + \sin(2n-1)x = \frac{1-\cos 2nx}{2\sin x},\ n \in \mathbb{Z}^+,\ x \neq k\pi$ where $k \in \mathbb{Z}$.

d. Hence or otherwise solve the equation $\sin x + \sin 3x = \cos x$ in the interval $0 < x < \pi$. [6]

The diagram shows a tangent, (TP) , to the circle with centre O and radius r . The size of $\mathrm{P\hat{O}A}$ is θ radians.



a. Find the area of triangle AOP in terms of r and θ . [1]

b. Find the area of triangle POT in terms of r and θ . [2]

c. Using your results from part (a) and part (b), show that $\sin\theta < \theta < \tan\theta$. [2]

The first three terms of a geometric sequence are $\sin x,\ \sin 2x$ and $4\sin x\cos^2 x,\ -\frac{\pi}{2} < x < \frac{\pi}{2}.$

(a) Find the common ratio r.

(b) Find the set of values of x for which the geometric series $\sin x + \sin 2x + 4\sin x\cos^2 x + \ldots$ converges.

Consider $x = \arccos\left(\frac{1}{4}\right),\ x > 0$.

(c) Show that the sum to infinity of this series is $\frac{\sqrt{15}}{2}$.

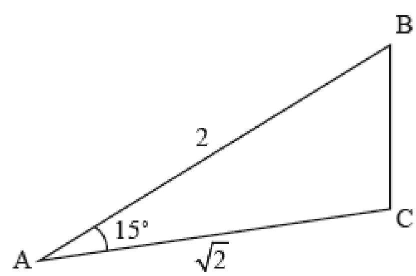
Solve the equation $\sin 2x - \cos 2x = 1 + \sin x - \cos x$ for $x \in [-\pi,\ \pi]$.

In triangle ABC, $\mathrm{BC} = \sqrt{3}$ cm, $\mathrm{A\hat{B}C} = \theta$ and $\mathrm{B\hat{C}A} = \frac{\pi}{3}$.

a. Show that length $\mathrm{AB} = \frac{3}{\sqrt{3}\cos\theta + \sin\theta}$. [4]

b. Given that AB has a minimum value, determine the value of θ for which this occurs. [4]

The following diagram shows the triangle ABC where $\mathrm{AB} = 2,\ \mathrm{AC} = \sqrt{2}$ and $\mathrm{B\hat{A}C} = 15°$.



a. Expand and simplify $\left(1 - \sqrt{3}\right)^2$. [1]

b. By writing $15°$ as $60° - 45°$ find the value of $\cos(15°)$. [3]

c. Find BC in the form $a + \sqrt{b}$ where $a,\ b \in \mathbb{Z}$. [4]

The triangle ABC is equilateral of side 3 cm. The point D lies on [BC] such that BD = 1 cm.

Find $\cos \mathrm{D\hat{A}C}$.

Let $f(x) = \frac{\sin 3x}{\sin x} - \frac{\cos 3x}{\cos x}$.

a. For what values of x does $f(x)$ not exist? [2]

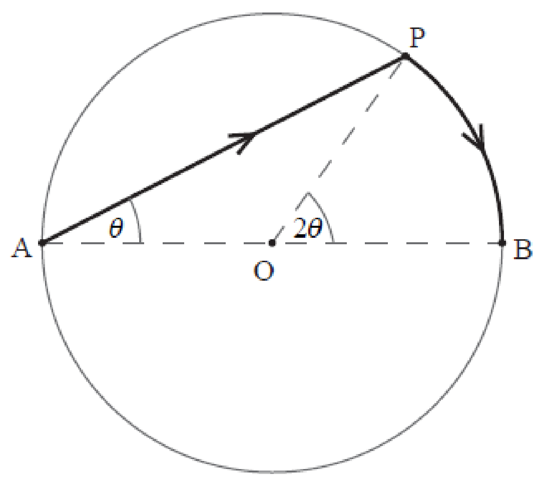
b. Simplify the expression $\frac{\sin 3x}{\sin x} - \frac{\cos 3x}{\cos x}$. [5]

A circular disc is cut into twelve sectors whose areas are in an arithmetic sequence.

The angle of the largest sector is twice the angle of the smallest sector.

Find the size of the angle of the smallest sector.

The diagram below shows a circular lake with centre O, diameter AB and radius 2 km.



Jorg needs to get from A to B as quickly as possible. He considers rowing to point P and then walking to point B. He can row at 3 km h^{-1} and walk at 6 km h^{-1}. Let $\text{P}\hat{\text{A}}\text{B} = \theta$ radians, and t be the time in hours taken by Jorg to travel from A to B.

a. Show that $t = \frac{2}{3}(2\cos\theta + \theta)$. [3]

b. Find the value of θ for which $\frac{\text{d}t}{\text{d}\theta} = 0$. [2]

c. What route should Jorg take to travel from A to B in the least amount of time? [3]

Give reasons for your answer.

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a. Write down the expansion of $(\cos\theta + \text{i}\sin\theta)^3$ in the form $a + \text{i}b$, where a and b are in terms of $\sin\theta$ and $\cos\theta$. [2]

b. Hence show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. [3]

c. Similarly show that $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$. [3]

d. **Hence** solve the equation $\cos 5\theta + \cos 3\theta + \cos\theta = 0$, where $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. [6]

e. By considering the solutions of the equation $\cos 5\theta = 0$, show that $\cos \frac{\pi}{10} = \sqrt{\frac{5+\sqrt{5}}{8}}$ and state the value of $\cos \frac{7\pi}{10}$. [8]

From a vertex of an equilateral triangle of side $2x$, a circular arc is drawn to divide the triangle into two regions, as shown in the diagram below.

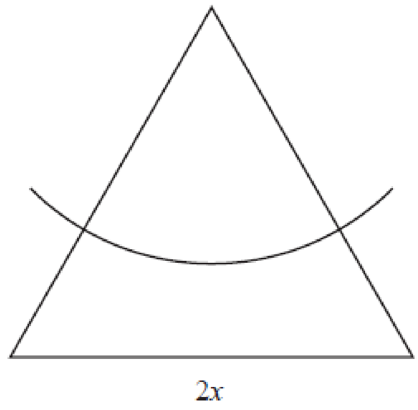


diagram not to scale

Given that the areas of the two regions are equal, find the radius of the arc in terms of x.

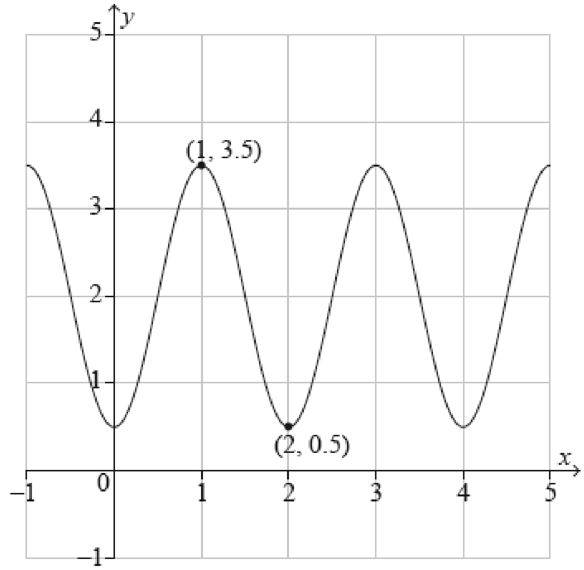
The angle θ lies in the first quadrant and $\cos\theta = \frac{1}{3}$.

a. Write down the value of $\sin\theta$. [1]

b. Find the value of $\tan 2\theta$. [2]

c. Find the value of $\cos\left(\frac{\theta}{2}\right)$, giving your answer in the form $\frac{\sqrt{a}}{b}$ where a , $b \in \mathbb{Z}^+$. [3]

The following diagram shows the curve $y = a\sin(b(x+c)) + d$, where a, b, c and d are all positive constants. The curve has a maximum point at $(1,\ 3.5)$ and a minimum point at $(2,\ 0.5)$.



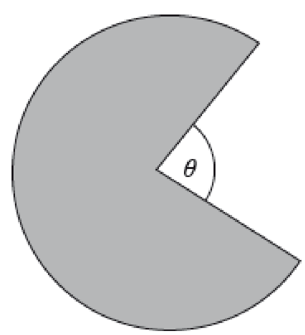
a. Write down the value of a and the value of d. [2]

b. Find the value of b. [2]

c. Find the smallest possible value of c, given $c > 0$. [2]

The logo, for a company that makes chocolate, is a sector of a circle of radius 2 cm, shown as shaded in the diagram. The area of the logo is $3\pi\ \mathrm{cm}^2$.

diagram not to scale

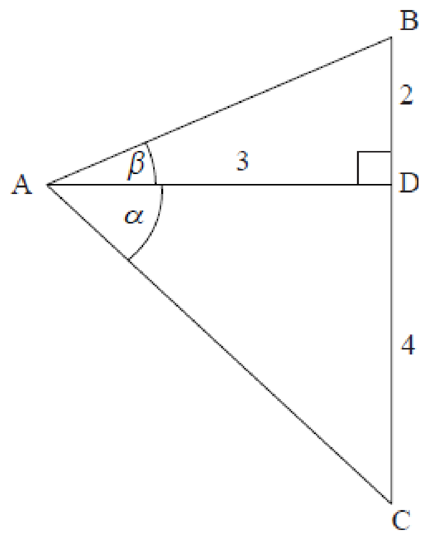


a. Find, in radians, the value of the angle θ, as indicated on the diagram. [3]

b. Find the total length of the perimeter of the logo. [2]

In the diagram below, AD is perpendicular to BC.

CD = 4, BD = 2 and AD = 3. $\mathrm{C\hat{A}D} = \alpha$ and $\mathrm{B\hat{A}D} = \beta$.



Find the exact value of $\cos(\alpha - \beta)$.

Consider the curve defined by the equation $x^2 + \sin y - xy = 0$.

a. Find the gradient of the tangent to the curve at the point $(\pi,\ \pi)$. [6]

b. Hence, show that $\tan\theta = \frac{1}{1+2\pi}$, where θ is the acute angle between the tangent to the curve at $(\pi,\ \pi)$ and the line $y = x$. [3]

Show that $\frac{\cos A + \sin A}{\cos A - \sin A} = \sec 2A + \tan 2A$.

(a) Prove the trigonometric identity $\sin(x+y)\sin(x-y) = \sin^2 x - \sin^2 y$.

(b) Given $f(x) = \sin(x+\frac{\pi}{6})\sin(x-\frac{\pi}{6}),\ x \in [0,\ \pi]$, find the range of f.

(c) Given $g(x) = \csc(x+\frac{\pi}{6})\csc(x-\frac{\pi}{6}),\ x \in [0,\ \pi]\ ,\ x \neq \frac{\pi}{6},\ x \neq \frac{5\pi}{6}$, find the range of g.

Consider the equation $\frac{\sqrt{3}-1}{\sin x} + \frac{\sqrt{3}+1}{\cos x} = 4\sqrt{2},\ 0 < x < \frac{\pi}{2}$. Given that $\sin\left(\frac{\pi}{12}\right) = \frac{\sqrt{6}-\sqrt{2}}{4}$ and $\cos\left(\frac{\pi}{12}\right) = \frac{\sqrt{6}+\sqrt{2}}{4}$

a. verify that $x = \frac{\pi}{12}$ is a solution to the equation; [3]

b. hence find the other solution to the equation for $0 < x < \frac{\pi}{2}$. [5]

Consider the functions $f(x) = \tan x,\ 0 \leq\ x < \frac{\pi}{2}$ and $g(x) = \frac{x+1}{x-1},\ x \in \mathbb{R},\ x \neq 1.$

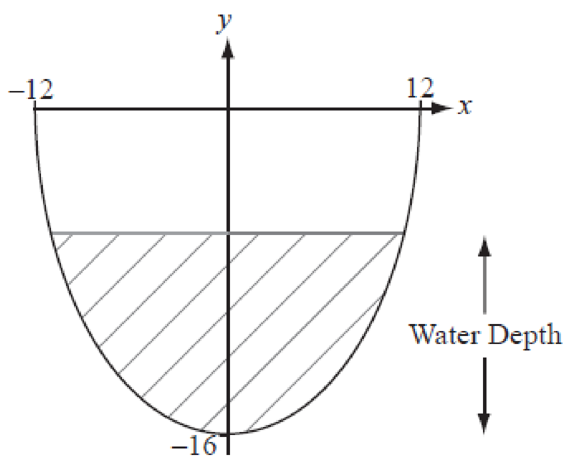
a. Find an expression for $g \circ f(x)$, stating its domain. [2]

b. Hence show that $g \circ f(x) = \frac{\sin x + \cos x}{\sin x - \cos x}$. [2]

c. Let $y = g \circ f(x)$, find an exact value for $\frac{dy}{dx}$ at the point on the graph of $y = g \circ f(x)$ where $x = \frac{\pi}{6}$, expressing your answer in the form $a + b\sqrt{3},\ a,\ b \in \mathbb{Z}$. [6]

d. Show that the area bounded by the graph of $y = g \circ f(x)$, the x-axis and the lines $x = 0$ and $x = \frac{\pi}{6}$ is $\ln\left(1 + \sqrt{3}\right)$. [6]

The diagram below shows the boundary of the cross-section of a water channel.



The equation that represents this boundary is $y = 16\sec\left(\frac{\pi x}{36}\right) - 32$ where x and y are both measured in cm.

The top of the channel is level with the ground and has a width of 24 cm. The maximum depth of the channel is 16 cm.

Find the width of the water surface in the channel when the water depth is 10 cm.

Give your answer in the form $a \arccos b$ where $a,\ b \in \mathbb{R}$.

In the triangle ABC, $\text{A}\hat{\text{B}}\text{C} = 90°$, $\text{AC} = \sqrt{2}$ and AB = BC + 1.

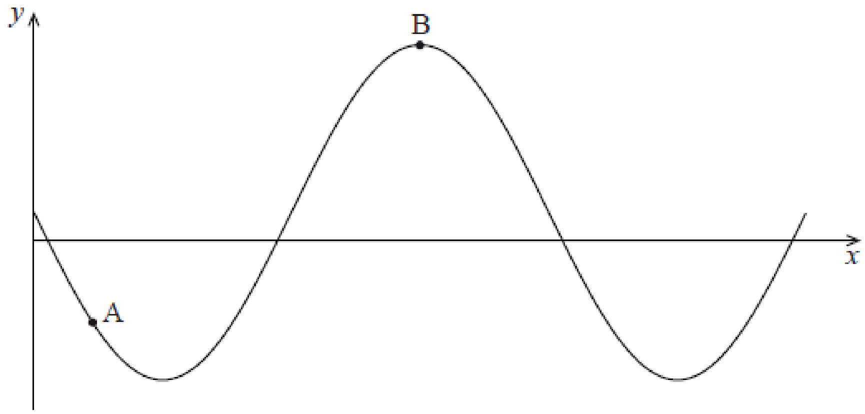
a. Show that $\cos \hat{A} - \sin \hat{A} = \frac{1}{\sqrt{2}}$. [3]

b. By squaring both sides of the equation in part (a), solve the equation to find the angles in the triangle. [8]

c. Apply Pythagoras' theorem in the triangle ABC to find BC, and hence show that $\sin \hat{A} = \frac{\sqrt{6} - \sqrt{2}}{4}$. [6]

d. Hence, or otherwise, calculate the length of the perpendicular from B to [AC]. [4]

The diagram below shows a curve with equation $y = 1 + k\sin x$, defined for $0 \leqslant x \leqslant 3\pi$.



The point A $\left(\frac{\pi}{6}, -2\right)$ lies on the curve and B$(a,\ b)$ is the maximum point.

(a) Show that $k = -6$.

(b) Hence, find the values of a and b .

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a. (i) Show that $(1 + \mathrm{i}\tan\theta)^n + (1 - \mathrm{i}\tan\theta)^n = \frac{2\cos n\theta}{\cos^n\theta}, \quad \cos\theta \neq 0.$ [10]

(ii) Hence verify that $\mathrm{i}\tan\frac{3\pi}{8}$ is a root of the equation $(1 + z)^4 + (1 - z)^4 = 0, \quad z \in \mathbb{C}.$

(iii) State another root of the equation $(1 + z)^4 + (1 - z)^4 = 0, \quad z \in \mathbb{C}.$

b. (i) Use the double angle identity $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$ to show that $\tan\frac{\pi}{8} = \sqrt{2} - 1.$ [13]

(ii) Show that $\cos 4x = 8\cos^4 x - 8\cos^2 x + 1.$

(iii) Hence find the value of $\int_0^{\frac{\pi}{8}} \frac{2\cos 4x}{\cos^2 x}\mathrm{d}x.$

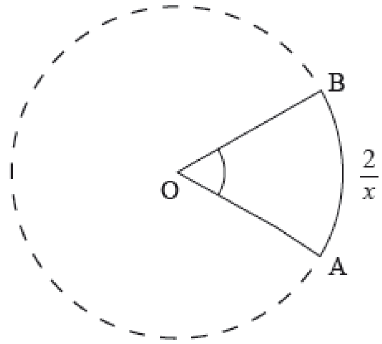
In the triangle PQR, PQ = 6 , PR = k and $\mathrm{P\hat{Q}R} = 30^\circ$.

a. For the case $k = 4$, find the two possible values of QR. [4]

b. Determine the values of k for which the conditions above define a unique triangle. [3]

The following diagram shows a sector of a circle where $\mathrm{A\hat{O}B} = x$ radians and the length of the arc $\mathrm{AB} = \frac{2}{x}\ \mathrm{cm}.$

Given that the area of the sector is $16\ \mathrm{cm}^2$, find the length of the arc AB.



Given that $\frac{\pi}{2} < \alpha < \pi$ and $\cos\alpha = -\frac{3}{4}$, find the value of $\sin 2\alpha$.

The vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ satisfy the equation $\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c} = \boldsymbol{0}$. Show that $\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{b} \times \boldsymbol{c} = \boldsymbol{c} \times \boldsymbol{a}$.

Consider the following functions:

$h(x) = \arctan(x),\ x \in \mathbb{R}$
$g(x) = \frac{1}{x}, x \in \mathbb{R},\ x \neq 0$

a. Sketch the graph of $y = h(x)$. [2]

b. Find an expression for the composite function $h \circ g(x)$ and state its domain. [2]

c. Given that $f(x) = h(x) + h \circ g(x)$, [7]

(i) find $f'(x)$ in simplified form;
(ii) show that $f(x) = \frac{\pi}{2}$ for $x > 0$.

d. Nigel states that f is an odd function and Tom argues that f is an even function. [3]

(i) State who is correct and justify your answer.
(ii) Hence find the value of $f(x)$ for $x < 0$.

a. Show that $\sin\left(\theta + \frac{\pi}{2}\right) = \cos\theta$. [1]

b. Consider $f(x) = \sin(ax)$ where a is a constant. Prove by mathematical induction that $f^{(n)}(x) = a^n \sin\left(ax + \frac{n\pi}{2}\right)$ where $n \in \mathbb{Z}^+$ and $f^{(n)}(x)$ represents the n^{th} derivative of $f(x)$. [7]

Solve the equation $\sec^2 x + 2\tan x = 0,\ 0 \leqslant x \leqslant 2\pi$.

(a) Show that $\sin 2nx = \sin((2n+1)x)\cos x - \cos((2n+1)x)\sin x$.

(b) **Hence** prove, by induction, that

$$\cos x + \cos 3x + \cos 5x + \ldots + \cos((2n-1)x) = \frac{\sin 2nx}{2\sin x},$$

for all $n \in \mathbb{Z}^+$, $\sin x \neq 0$.

(c) Solve the equation $\cos x + \cos 3x = \frac{1}{2}$, $0 < x < \pi$.

A triangle has sides of length (n^2+n+1), $(2n+1)$ and (n^2-1) where $n > 1$.

(a) Explain why the side (n^2+n+1) must be the longest side of the triangle.

(b) Show that the largest angle, θ, of the triangle is 120°.

Given that $\sin x + \cos x = \frac{2}{3}$, find $\cos 4x$.

Find all solutions to the equation $\tan x + \tan 2x = 0$ where $0^\circ \leq x < 360^\circ$.

The diagram below shows two straight lines intersecting at O and two circles, each with centre O. The outer circle has radius *R* and the inner circle has radius *r* .

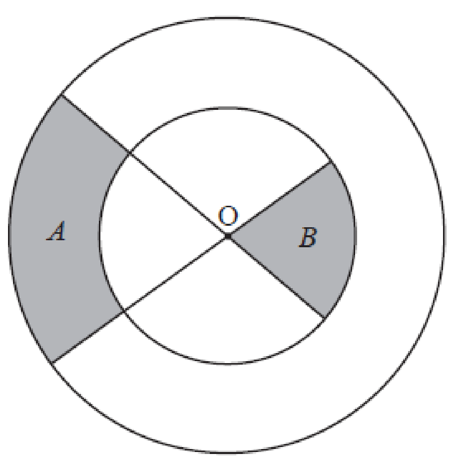


diagram not to scale

Consider the shaded regions with areas *A* and *B* . Given that $A : B = 2 : 1$, find the **exact** value of the ratio $R : r$.

Consider the following system of equations:

$$x + y + z = 1$$

$$2x + 3y + z = 3$$

$$x + 3y - z = \lambda$$

where $\lambda \in \mathbb{R}$.

a. Show that this system does not have a unique solution for any value of λ . [4]

b. (i) Determine the value of λ for which the system is consistent. [4]

(ii) For this value of λ , find the general solution of the system.

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a. Sketch the graph of $y = \left|\cos\left(\frac{x}{4}\right)\right|$ for $0 \leqslant x \leqslant 8\pi$. [2]

b. Solve $\left|\cos\left(\frac{x}{4}\right)\right| = \frac{1}{2}$ for $0 \leqslant x \leqslant 8\pi$. [3]

In triangle ABC, AB = 9 cm , AC = 12 cm , and $\hat{B}$ is twice the size of $\hat{C}$.

Find the cosine of $\hat{C}$.

Consider $w = 2\left(\cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3}\right)$

These four points form the vertices of a quadrilateral, *Q*.

a.i. Express w^2 and w^3 in modulus-argument form. [3]

a.ii. Sketch on an Argand diagram the points represented by w^0 , w^1 , w^2 and w^3. [2]

b. Show that the area of the quadrilateral *Q* is $\frac{21\sqrt{3}}{2}$. [3]

c. Let $z = 2\left(\cos\frac{\pi}{n} + \mathrm{i}\sin\frac{\pi}{n}\right),\ n \in \mathbb{Z}^+$. The points represented on an Argand diagram by $z^0,\ z^1,\ z^2,\ \ldots,\ z^n$ form the vertices of a polygon P_n. [6]

Show that the area of the polygon P_n can be expressed in the form $a\left(b^n - 1\right)\sin\frac{\pi}{n}$, where $a,\ b \in \mathbb{R}$.

Let $a = \sin b,\ 0 < b < \frac{\pi}{2}$.

Find, in terms of b, the solutions of $\sin 2x = -a,\ 0 \leqslant x \leqslant \pi$.

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a. Find all values of x for $0.1 \leqslant x \leqslant 1$ such that $\sin(\pi x^{-1}) = 0$. [2]

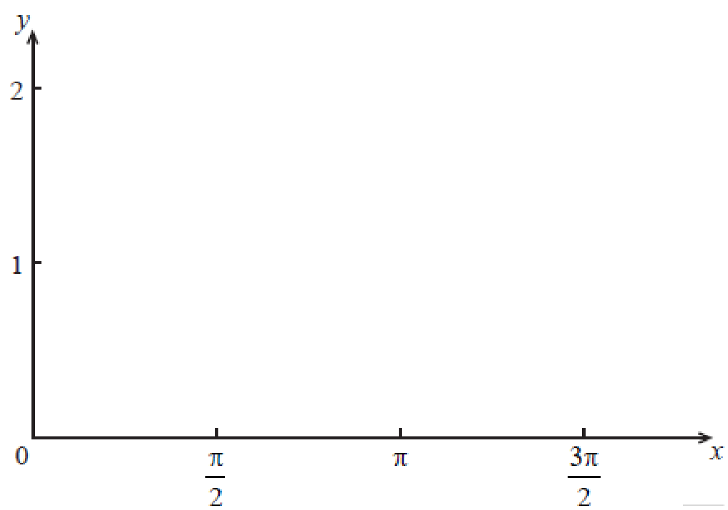
b. Find $\int_{\frac{1}{n+1}}^{\frac{1}{n}} \pi x^{-2} \sin(\pi x^{-1}) \mathrm{d}x$, showing that it takes different integer values when n is even and when n is odd. [3]

c. Evaluate $\int_{0.1}^{1} |\pi x^{-2} \sin(\pi x^{-1})| \,\mathrm{d}x$. [2]

If x satisfies the equation $\sin\left(x + \frac{\pi}{3}\right) = 2 \sin x \sin\left(\frac{\pi}{3}\right)$, show that $11 \tan x = a + b\sqrt{3}$, where $a, b \in \mathbb{Z}^+$.

Given that $f(x) = 1 + \sin x,\ 0 \leqslant x \leqslant \frac{3\pi}{2}$,

a. sketch the graph of f; [1]



b. show that $(f(x))^2 = \frac{3}{2} + 2 \sin x - \frac{1}{2} \cos 2x$; [1]

c. find the volume of the solid formed when the graph of f is rotated through 2π radians about the x-axis. [4]

The function f is defined on the domain $\left[0, \frac{3\pi}{2}\right]$ by $f(x) = e^{-x} \cos x$.

a. State the two zeros of f. [1]

b. Sketch the graph of f. [1]

c. The region bounded by the graph, the x-axis and the y-axis is denoted by A and the region bounded by the graph and the x-axis is denoted by B. Show that the ratio of the area of A to the area of B is [7]

$$\frac{e^{\pi}\left(e^{\frac{\pi}{2}}+1\right)}{e^{\pi}+1}.$$

The function f is defined by $f(x)=\frac{1}{4x^2-4x+5}$.

a. Express $4x^2-4x+5$ in the form $a(x-h)^2+k$ where $a, h, k \in \mathbb{Q}$. [2]

b. The graph of $y=x^2$ is transformed onto the graph of $y=4x^2-4x+5$. Describe a sequence of transformations that does this, making the order of transformations clear. [3]

c. Sketch the graph of $y=f(x)$. [2]

d. Find the range of f. [2]

e. By using a suitable substitution show that $\int f(x)\mathrm{d}x=\frac{1}{4}\int\frac{1}{u^2+1}\mathrm{d}u$. [3]

f. Prove that $\int_1^{3.5}\frac{1}{4x^2-4x+5}\mathrm{d}x=\frac{\pi}{16}$. [7]

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a. (i) Sketch the graphs of $y=\sin x$ and $y=\sin 2x$, on the same set of axes, for $0\leqslant x\leqslant\frac{\pi}{2}$. [9]

(ii) Find the x-coordinates of the points of intersection of the graphs in the domain $0\leqslant x\leqslant\frac{\pi}{2}$.

(iii) Find the area enclosed by the graphs.

b. Find the value of $\int_0^1\sqrt{\frac{x}{4-x}}\mathrm{d}x$ using the substitution $x=4\sin^2\theta$. [8]

c. The increasing function f satisfies $f(0)=0$ and $f(a)=b$, where $a>0$ and $b>0$. [8]

(i) By reference to a sketch, show that $\int_0^a f(x)\mathrm{d}x=ab-\int_0^b f^{-1}(x)\mathrm{d}x$.

(ii) **Hence** find the value of $\int_0^2\arcsin\left(\frac{x}{4}\right)\mathrm{d}x$.

(a) Show that $\arctan\left(\frac{1}{2}\right)+\arctan\left(\frac{1}{3}\right)=\frac{\pi}{4}$.

(b) Hence, or otherwise, find the value of $\arctan(2)+\arctan(3)$.

(a) Sketch the curve $f(x)=\sin 2x$, $0\leqslant x\leqslant\pi$.

(b) Hence sketch on a separate diagram the graph of $g(x)=\csc 2x$, $0\leqslant x\leqslant\pi$, clearly stating the coordinates of any local maximum or minimum points and the equations of any asymptotes.

(c) Show that $\tan x+\cot x\equiv 2\csc 2x$.

(d) Hence or otherwise, find the coordinates of the local maximum and local minimum points on the graph of $y = \tan 2x + \cot 2x$, $0 \leqslant x \leqslant \frac{\pi}{2}$

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(e) Find the solution of the equation $\csc 2x = 1.5 \tan x - 0.5$, $0 \leqslant x \leqslant \frac{\pi}{2}$.